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# Symbolic operational images and decomposition formulas for hypergeometric functions

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ABSTRACT

Based upon the classical derivative and integral operators we introduce a new operator which allows the derivation of new symbolic operational images for hypergeometric functions. By means of these symbolic operational images a number of decomposition formulas involving quadruple series are then found. Other closely-related results are also considered.



## 1. Introduction, notations and definitions

Hypergeometric functions in one and more variables occur naturally in a wide variety of problems (see, for details [5] and [15]). In particular, the generalized hypergeometric function ${}_AF_B$ with $A$ numerator and $B$ denominator parameters

$$ {}_AF_B\big[(a_A);(b_B);x\big]=\sum_{m=0}^{\infty}\frac{((a_A))_m}{((b_B))_m}\frac{x^m}{m!}, \tag{1.1} $$

and its special cases the Gaussian hypergeometric function

$$ {}_2F_1[\alpha,\beta;\gamma;x]=\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\frac{x^m}{m!}, \tag{1.2} $$

and the Kummerian function

$$ {}_1F_1[\alpha;\gamma;x]=\sum_{m=0}^{\infty}\frac{(\alpha)_m}{(\gamma)_m}\frac{x^m}{m!}, \tag{1.3} $$

where $(a_A)=a_1,a_2,\dots,a_A$ and $(a)_n$ denotes the Pochhammmer symbol given by

$$ (a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=a(a+1)\cdots(a+n-1)\quad\text{and}\quad(a)_0=1, $$

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arise frequently in various physical, quantum chemical and statistical applications (cf. [6, Chapters 7 and 8] and [18, Section 1.7]). Appell [18, Section 1.3] defined the following four hypergeometric series in two variable:

$$F_1[a,b,\acute{b};c;x,y]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(\acute{b})_n}{(c)_{m+n}}\frac{x^m y^n}{m!n!},\tag{1.4}$$

$$F_2[a,b,\acute{b};c,\acute{c};x,y]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(\acute{b})_n}{(c)_m(\acute{c})_n}\frac{x^m y^n}{m!n!},\tag{1.5}$$

$$F_3[a,\acute{a},b,\acute{b};c;x,y]=\sum_{m,n=0}^{\infty}\frac{(a)_m(\acute{a})_n(b)_m(\acute{b})_n}{(c)_{m+n}}\frac{x^m y^n}{m!n!},\tag{1.6}$$

$$F_4[a,b;c,\acute{c};x,y]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m(\acute{c})_n}\frac{x^m y^n}{m!n!}.\tag{1.7}$$

Kámpe de Fériet [18, p. 27, (28)] further generalized the function ${}_AF_B$ and the four Appell series $F_1, F_2, F_3$ and $F_4$ to general hypergeometric series in two variables and defined his double hypergeometric series as follows:

$$\begin{aligned}F^{(2)}[x,y]&=F^{A:B;\acute{B}}_{E:G;\acute{G}}\left[\begin{matrix}(a_A):(b_B);(\acute{b}_{\acute{B}});\\(e_E):(g_G);(\acute{g}_{\acute{G}});\end{matrix}\quad x,y\right]\\&=\sum_{m,n=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m+n}\prod_{j=1}^{B}(b_j)_m\prod_{j=1}^{\acute{B}}(\acute{b}_j)_n}{\prod_{j=1}^{E}(e_j)_{m+n}\prod_{j=1}^{G}(g_j)_m\prod_{j=1}^{\acute{G}}(\acute{g}_j)_n}\frac{x^m y^n}{m!n!}.\end{aligned}\tag{1.8}$$

Srivastava [18, p. 44, (14) and (15)] defined a general triple hypergeometric series $F^{(3)}[x,y,z]$ by

$$\begin{aligned}F^{(3)}[x,y,z]&=F^{(3)}\left[\begin{matrix}(a_A)::(b_B);(\acute{b}_{\acute{B}});(\acute{\acute{b}}_{\acute{\acute{B}}}):(c_C);(\acute{c}_{\acute{C}});(\acute{\acute{c}}_{\acute{\acute{C}}});\\(e_E)::(g_G);(\acute{g}_{\acute{G}});(\acute{\acute{g}}_{\acute{\acute{G}}}):(h_H);(\acute{h}_{\acute{H}});(\acute{\acute{h}}_{\acute{\acute{H}}});\end{matrix}\quad x,y,z\right]\\&=\sum_{m,n,p=0}^{\infty}\Lambda(m,n,p)\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\end{aligned}\tag{1.9}$$

where, for convenience,

$$\begin{aligned}\Lambda(m,n,p)=&\frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\prod_{j=1}^{B}(b_j)_{m+n}\prod_{j=1}^{\acute{B}}(\acute{b}_j)_{n+p}\prod_{j=1}^{\acute{\acute{B}}}(\acute{\acute{b}}_j)_{p+m}}{\prod_{j=1}^{E}(e_j)_{m+n+p}\prod_{j=1}^{G}(g_j)_{m+n}\prod_{j=1}^{\acute{G}}(\acute{g}_j)_{n+p}\prod_{j=1}^{\acute{\acute{G}}}(\acute{\acute{g}}_j)_{p+m}}\\&\times\frac{\prod_{j=1}^{C}(c_j)_m\prod_{j=1}^{\acute{C}}(\acute{c}_j)_n\prod_{j=1}^{\acute{\acute{C}}}(\acute{\acute{c}}_j)_p}{\prod_{j=1}^{H}(h_j)_m\prod_{j=1}^{\acute{H}}(\acute{h}_j)_n\prod_{j=1}^{\acute{\acute{H}}}(\acute{\acute{h}}_j)_p}.\end{aligned}\tag{1.10}$$

Note that, the general triple series in (1.9) is a unification of Lauricella's 14 triple hypergeometric series [18, Sections 1.4 and 1.5] $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$, $F_D^{(3)}$, $F_E$, $F_F$, $F_G$, $F_K$, $F_M$, $F_N$, $F_P$, $F_R$, $F_S$ and $F_T$ and Srivastava's triple series [18, Section 1.5] $H_A$, $H_B$ and $H_C$. Exton [6, pp. 78–79] introduced 21 complete hypergeometric functions $K_1, K_2, \dots, K_{21}$ of four variables. In [14] Sharma and Parihar introduced 83 complete hypergeometric functions $F_1^{(4)}, F_2^{(4)}, \dots, F_{83}^{(4)}$ of four variables. It is remarkable that out of these 83 functions, the following 19 functions had already appeared in the work of Exton [6] in the different notations:

$$\begin{gathered}F_9^{(4)}=K_1,\quad F_1^{(4)}=K_2,\quad F_{38}^{(4)}=K_3,\quad F_{10}^{(4)}=K_4,\quad F_2^{(4)}=K_5,\quad F_{59}^{(4)}=K_6,\quad F_{39}^{(4)}=K_7,\\F_{11}^{(4)}=K_8,\quad F_{12}^{(4)}=K_9,\quad F_3^{(4)}=K_{10},\quad F_{60}^{(4)}=K_{11},\quad F_{40}^{(4)}=K_{12},\quad F_{13}^{(4)}=K_{13},\quad F_{77}^{(4)}=K_{14},\\F_{78}^{(4)}=K_{15},\quad F_{79}^{(4)}=K_{16},\quad F_{82}^{(4)}=K_{19},\quad F_{81}^{(4)}=K_{20},\quad F_{83}^{(4)}=K_{21}.\end{gathered}$$

Each quadruple hypergeometric function is of the form

$$F^{(4)}(.)=\sum_{m,n,p,q=0}^{\infty}\Delta(m,n,p,q)\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}$$

where $\Delta(m,n,p,q)$ is a certain sequence of complex parameters and there are twelve parameters in each function $F^{(4)}(.)$ (four $a$'s, four $b$'s and four $c$'s). The 1st, 2nd, 3rd and 4th parameter of each kind is associated with the integers $m$, $n$, $p$ and $q$ respectively. A repeated parameter in $F^{(4)}(.)$ indicates the presence of a term with double summations in $\Delta(m,n,p,q)$. Thus $F^{(4)}(a_1,a_1,a_2,a_3)$ means $(a_1)_{m+n}(a_2)_p(a_3)_q$, $F^{(4)}(a_1,a_2,a_2,a_2)$ means $(a_1)_m(a_2)_{n+p+q}$ and $F^{(4)}(a_1,a_1,a_1,a_1)$ means $(a_1)_{m+n+p+q}$. The numerator and the denominator parameters are separated by means of a semicolon. The conventions and notations of the quadruple hypergeometric functions mentioned above are exemplified here only by the functions $K_1$ and $F_4^{(4)}$ as follows

$$K_1(a,a,a,a,b,b,b,c;d,e_1,e_2,d;x,y,z,t)=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{m+n+p+q}(b)_{m+n+p}(c)_q}{(d)_{m+q}(e_1)_n(e_2)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!},$$

$$F_4^{(4)}(a,a,a,a,b,b,c,b;e_1,e_2,e_3,e_4;x,y,z,t)=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{m+n+p+q}(b)_{m+n+q}(c)_p}{(e_1)_m(e_2)_n(e_3)_p(e_4)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}.$$

Similarly, the definitions of the rest of the quadruple functions can be indicated.

This work aims at introducing symbolic operational images (or representations) for hypergeometric functions of one, two and three variables and show how to exploit such operational images in order to establish decomposition formulas involving the above mentioned hypergeometric functions of one, two, three and four variables. In this work we apply the concept of the right-Riemann–Liouville fractional derivative to obtain operational images and decomposition formulas for hypergeometric functions, which is the most suitable technique among various approaches for defining a fractional derivative. Indeed, in this work we build up the right Riemann–Liouville fractional derivative operator, which plays the role of augmenting parameters in the hypergeometric functions involved (see [19, Chapters 4 and 5]). In particular, we will deal with operational definitions ruled by the operators $\hat{D}_x$ and $\hat{D}_x^{-1}$ where $\hat{D}_x$ denotes the derivative operator and $\hat{D}_x^{-1}$ defines the inverse of the derivative and once acting on unity yields

$$\hat{D}_x^{-m}1=\frac{x^m}{m!},\quad m\in N\cup\{0\}.$$

It is evident that $D_x^{-1}$ is essentially an integral operator and the lower limit has assumed to be zero (for more details see [4]). Also, we have

$$\hat{D}_x^{-m}\hat{D}_x^{-n}=\hat{D}_x^{-m-n},\quad\text{and}\quad\hat{D}_x^{m}\hat{D}_x^{n}=\hat{D}_x^{m+n}.$$

The following two formulas are the well-known consequences of the derivative operator $\hat{D}_x$ and the integral operator $\hat{D}_x^{-1}$ [11]

$$\hat{D}_x^{m}x^{\lambda}=\frac{\Gamma(\lambda-1)}{\Gamma(\lambda-m+1)}x^{\lambda-m},\tag{1.11}$$

$$\hat{D}_x^{-m}x^{\lambda}=\frac{\Gamma(\lambda-1)}{\Gamma(\lambda+m+1)}x^{\lambda+m},\tag{1.12}$$

$m\in N\cup\{0\}$, $\lambda\in C/\{-1,-2,\ldots\}$. Based on the operational relations (1.11) and (1.12) the action of the set of operators

$$\{\hat{D}_t^{m}\hat{D}_u^{-m}\}\quad(m=0,1,2,3,\ldots),$$

on the powers

$$\{t^{\beta+m-1}u^{\gamma-1}\}$$
$$\big(Re(\beta)>0,\ Re(\gamma)>0\big),$$

is given by

$$\hat{D}_t^{m}\hat{D}_u^{-m}\{t^{\beta+m-1}u^{\gamma-1}\}=\frac{(\beta)_m}{(\gamma)_m}\{t^{\beta-1}u^{\gamma+m-1}\}.\tag{1.13}$$

**Lemma 1.** *Let $Re(\beta)>0$ and $Re(\gamma)>0$, then*

$$\{t^{\beta-1}u^{\gamma-1}\}{}_2F_1[\alpha,\beta;\gamma;x]=\big(1-x\hat{D}_t u^{-1}\hat{D}_u^{-1}t\big)^{-\alpha}\{t^{\beta-1}u^{\gamma-1}\}.\tag{1.14}$$

**Proof.** Denote, for convenience, the right-hand side of assertion (1.14) by $I$. Then as a consequence of the binomial theorem, it is easily seen that:

$$I=\sum_{m=0}^{\infty}\frac{(\alpha)_m x^m}{m!u^m}\hat{D}_t^m\hat{D}_u^{-m}\{t^{\beta+m-1}u^{\gamma-1}\}. \tag{1.15}$$

Upon using (1.13) and considering the definition (1.2), we are led finally to the left-hand side of the assertion (1.14). □

Equivalently, we may write (1.14) in the form:

$$\left(1-\hat{D}_t\hat{D}_x^{-1}t\right)^{-\alpha}\{t^{\beta-1}x^{\gamma-1}\}=\{t^{\beta-1}x^{\gamma-1}\}{}_2F_1[\alpha,\beta;\gamma;x]. \tag{1.16}$$

By introducing the constricted notation

$$\hat{D}^m[t_{(A)};u_{(B)}]=\prod_{j=1}^{A}\left\{\hat{D}_{t_j}^m\left\{\prod_{i=1}^{B}\left(u_i^{-1}\hat{D}_{u_i}^{-1}\right)^m\right\}t_j^m\right\}\quad(m=0,1,2,3,\ldots), \tag{1.17}$$

it is easily verified that

$$\begin{aligned}&\hat{D}^m[t_{(A)};u_{(B)}]\left\{\prod_{j=1}^{A}\left(t_j^{a_j-1}\right)\prod_{j=1}^{B}\left(u_j^{b_j-1}\right)\right\}\\&\quad=\left\{\prod_{j=1}^{A}\left(t_j^{a_j-1}\right)\prod_{j=1}^{B}\left(u_j^{b_j-1}\right)\right\}\frac{\prod_{j=1}^{A}(a_j)_m}{\prod_{j=1}^{B}(b_j)_m}\end{aligned} \tag{1.18}$$

and

$$\hat{D}^{m+n}[t_{(A)};u_{(B)}]=\hat{D}^m[t_{(A)};u_{(B)}]\hat{D}^n[t_{(A)};u_{(B)}]=\hat{D}^{n+m}[t_{(A)};u_{(B)}]. \tag{1.19}$$

In particular, we have

$$\hat{D}^m[t_{(1)};u_{(0)}]\{t^{a-1}\}=\{t^{a-1}\}(a)_m,\qquad \hat{D}^m[t_{(0)};u_{(1)}]\{u^{b-1}\}=\{u^{b-1}\}\frac{1}{(b)_m},$$

and

$$\hat{D}^m[t_{(1)};u_{(1)}]\{t^{a-1}u^{b-1}\}=\{t^{a-1}u^{b-1}\}\frac{(a)_m}{(b)_m}.$$

Therefore, we may write (1.14) in the equivalent form

$$\left(1-x\hat{D}[t_{(1)};u_{(1)}]\right)^{-\alpha}\{t^{\beta-1}u^{\gamma-1}\}=\{t^{\beta-1}u^{\gamma-1}\}{}_2F_1[\alpha,\beta;\gamma;x]. \tag{1.20}$$

Further, for the powers

$$\left\{\prod_{j=1}^{A}\left(t_j^{a_j-1}\right)\prod_{j=1}^{B}\left(u_j^{b_j-1}\right)\prod_{j=1}^{C}\left(p_j^{c_j-1}\right)\prod_{j=1}^{G}\left(q_j^{g_j-1}\right)\right\}, \tag{1.21}$$

we have

$$\begin{aligned}&\left(1-\hat{D}[t_{(A)};u_{(B)}]\right)\left(1-\hat{D}[p_{(C)};q_{(G)}]\right)\\&\quad=\left(1-\hat{D}[t_{(A)};u_{(B)}]-\hat{D}[p_{(C)};q_{(G)}]+\hat{D}[t_{(A)};u_{(B)}]\hat{D}[p_{(C)};q_{(G)}]\right).\end{aligned} \tag{1.22}$$

To verify (1.22) apply both sides of (1.22) to (1.21) and use the operator (1.17) to check that they are equal. Note that, analogous results hold true for products of the types

$$\left(1-\hat{D}[t_{(A)};u_{(B)}]\right)\left(1-\hat{D}[p_{(C)};q_{(G)}]\right)\left(1-\hat{D}[\acute{t}_{(\acute{A})};\acute{u}_{(\acute{B})}]\right)\left(1-\hat{D}[\acute{p}_{(\acute{C})};\acute{q}_{(\acute{G})}]\right),$$

$$\left(1-\hat{D}[t_{(A)};u_{(B)}]-\hat{D}[\acute{t}_{(\acute{A})};\acute{u}_{(\acute{B})}]\right)\left(1-\hat{D}[p_{(C)};q_{(G)}]-\hat{D}[\acute{p}_{(\acute{C})};\acute{q}_{(\acute{G})}]\right),$$

and

$$\left(1-\hat{D}[t_{(A)};u_{(B)}]\right)\left(1-\hat{D}[\acute{t}_{(\acute{A})};\acute{u}_{(\acute{B})}]\right)\left(1-\hat{D}[p_{(C)};q_{(G)}]-\hat{D}[\acute{p}_{(\acute{C})};\acute{q}_{(\acute{G})}]\right). \tag{1.23}$$

Moreover, the multinomial expansion of algebra has its analogue the multinomial expansion:

$$\begin{aligned}&\left(1-\hat{D}[t_{1(A_1)};u_{1(B_1)}]-\cdots-\hat{D}[t_{n(A_n)};u_{n(B_n)}]\right)^{-\alpha}\\&\quad=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}}{m_1!\cdots m_n!}\hat{D}^{m_1}[t_{1(A_1)};u_{1(B_1)}]\cdots\hat{D}^{m_n}[t_{n(A_n)};u_{n(B_n)}].\end{aligned} \tag{1.24}$$

According to (1.24) and (1.18), we introduce the following symbolic operational image for the generalized hypergeometric series ${}_AF_B$ defined by (1.1).

**Lemma 2.** *Let $Re(a_j) > 0$ $(j = 1, 2, \dots, A)$ and $Re(b_i) > 0$ $(i = 1, 2, \dots, B)$, then*

$$\big(1 - x\hat{D}[p_{(A)}; q_{(B)}]\big)^{-\alpha} \left\{ \prod_{j=1}^{A} \big(p_j^{a_j - 1}\big) \prod_{j=1}^{B} \big(q_j^{b_j - 1}\big) \right\} = \left\{ \prod_{j=1}^{A} \big(p_j^{a_j - 1}\big) \prod_{j=1}^{B} \big(q_j^{b_j - 1}\big) \right\} {}_{1+A}F_B\big[\alpha, (a_A); (b_B); x\big]. \tag{1.25}$$

**Proof.** As a consequence of (1.24), it is easily seen that

$$\big(1 - x\hat{D}[p_{(A)}; q_{(B)}]\big)^{-\alpha} \left\{ \prod_{j=1}^{A} \big(p_j^{a_j - 1}\big) \prod_{j=1}^{B} \big(q_j^{b_j - 1}\big) \right\} = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{m!} x^m \hat{D}^m [p_{(A)}; q_{(B)}] \left\{ \prod_{j=1}^{A} \big(p_j^{a_j - 1}\big) \prod_{j=1}^{B} \big(q_j^{b_j - 1}\big) \right\}.$$

The result now follows from (1.18) and the definition (1.1). □

Note that a fairly wide variety of special functions can be represented in terms of the hypergeometric series in (1.1) and (1.2) (see e.g. [13] and [18]). According to these facts a number of symbolic operational images for certain functions and polynomials are obtainable by appropriate specialization of the parameters in the operational images (1.14) and (1.25). For example, in view of the following hypergeometric representations

$$P_\nu(x) = {}_2F_1\left[-\nu, \nu + 1; 1; \frac{1 - x}{2}\right], \tag{1.26}$$

and

$$f_n\big[(a_A); (b_B); x\big] = {}_{2+A}F_{B+2}\left[\begin{matrix} -n, n+1, (a_A); \\ 1, 1/2, (b_B); \end{matrix} \; x\right], \tag{1.27}$$

where $P_\nu(x)$ and $f_n$ are the Legendre polynomials and Sister Celine Laguerre polynomials (see [13, p. 290]), and the operational images (1.14) and (1.25), we can state

$$\left[1 - \left(\frac{1 - x}{2}\right) \hat{D}[t_{(1)}; u_{(1)}]\right]^{\nu} \big\{t^\nu\big\} = \big\{t^\nu\big\} P_\nu(x), \tag{1.28}$$

and

$$\big(1 - x\hat{D}[p_{(A+1)}; q_{(B+2)}]\big)^{n} \left\{ \prod_{j=1}^{A} \big(p_j^{a_j - 1}\big) \prod_{j=1}^{B} \big(q_j^{b_j - 1}\big) \cdot p_{A+1}^{n} \cdot q_{B+2}^{-\frac{1}{2}} \right\} = \left\{ \prod_{j=1}^{A} \big(p_j^{a_j - 1}\big) \prod_{j=1}^{B} \big(q_j^{b_j - 1}\big) \cdot p_{A+1}^{n} \cdot q_{B+2}^{-\frac{1}{2}} \right\} f_n\big[(a_A); (b_B); x\big]. \tag{1.29}$$

Next, directly, from the definitions of the operator $\hat{D}[t_{(A)}; u_{(B)}]$ (cf. (1.17)) and the definition of Srivastava triple hypergeometric series $F^{(3)}[x, y, z]$ (cf. (1.9)), by exploiting the same procedure leading to the symbolic operational images (1.14) and (1.25), we can establish the following operational image for the function $F^{(3)}[x, y, z]$.

**Lemma 3.** *Let*

$$Re(a_A) > 0, \quad Re(e_E) > 0, \quad Re(b_B) > 0, \quad Re(\acute{b}_{\acute{B}}) > 0, \quad Re(\acute{\acute{b}}_{\acute{\acute{B}}}) > 0,$$

$$Re(g_G) > 0, \quad Re(\acute{g}_{\acute{G}}) > 0, \quad Re(\acute{\acute{g}}_{\acute{\acute{G}}}) > 0, \quad Re(c_C) > 0, \quad Re(\acute{c}_{\acute{C}}) > 0, \quad Re(\acute{\acute{c}}_{\acute{\acute{C}}}) > 0,$$

$$Re(h_H) > 0, \quad Re(\acute{h}_{\acute{H}}) > 0, \quad Re(\acute{\acute{h}}_{\acute{\acute{H}}}) > 0,$$

*where $Re(a_A)$ abbreviates the array of $A$ parameters $Re(a_1), Re(a_2), \dots, Re(a_A)$, with similar interpretations for $Re(e_E)$, $Re(b_B)$, et cetera. Let, for convenience,*

$$\hat{M} = \hat{D}[t_{(A)}; u_{(E)}] \quad \hat{D}[p_{(B)}; q_{(G)}] \quad \hat{D}[\acute{\acute{p}}_{(\acute{\acute{B}})}; \acute{\acute{q}}_{(\acute{\acute{G}})}] \quad \hat{D}[r_{(C)}; k_{(H)}], \tag{1.30}$$

$$\hat{N} = \hat{D}[t_{(A)}; u_{(E)}] \quad \hat{D}[p_{(B)}; q_{(G)}] \quad \hat{D}[\acute{p}_{(\acute{B})}; \acute{q}_{(\acute{G})}] \quad \hat{D}[\acute{r}_{(\acute{C})}; \acute{k}_{(\acute{H})}], \tag{1.31}$$

$$\hat{R} = \hat{D}[t_{(A)}; u_{(E)}] \quad \hat{D}[\acute{p}_{(\acute{B})}; \acute{q}_{(\acute{G})}] \quad \hat{D}[\acute{\acute{p}}_{(\acute{\acute{B}})}; \acute{\acute{q}}_{(\acute{\acute{G}})}] \quad \hat{D}[\acute{\acute{r}}_{(\acute{\acute{C}})}; \acute{\acute{k}}_{(\acute{\acute{H}})}], \tag{1.32}$$

$$\begin{aligned}
&\prod\big(t^a, u^e, p^b, \acute{p}^{\acute{b}}, \acute{\acute{p}}^{\acute{\acute{b}}}, q^g, \acute{q}^{\acute{g}}, \acute{\acute{q}}^{\acute{\acute{g}}}, r^c, \acute{r}^{\acute{c}}, \acute{\acute{r}}^{\acute{\acute{c}}} k^h, \acute{k}^{\acute{h}}, \acute{\acute{k}}^{\acute{\acute{h}}}\big) \\
&\quad = \Bigg\{ \prod_{j=1}^{A}\big(t_j^{a_j-1}\big) \prod_{j=1}^{E}\big(u_j^{e_j-1}\big) \prod_{j=1}^{B}\big(p_j^{b_j-1}\big) \prod_{j=1}^{\acute{B}}\big(\acute{p}_j^{\acute{b}_j-1}\big) \prod_{j=1}^{\acute{\acute{B}}}\big(\acute{\acute{p}}_j^{\acute{\acute{b}}_j-1}\big) \prod_{j=1}^{G}\big(q_j^{g_j-1}\big) \prod_{j=1}^{\acute{G}}\big(\acute{q}_j^{\acute{g}_j-1}\big) \\
&\qquad \times \prod_{j=1}^{\acute{\acute{G}}}\big(\acute{\acute{q}}_j^{\acute{\acute{g}}_j-1}\big) \prod_{j=1}^{C}\big(r_j^{c_j-1}\big) \prod_{j=1}^{\acute{C}}\big(\acute{r}_j^{\acute{c}_j-1}\big) \prod_{j=1}^{\acute{\acute{C}}}\big(\acute{\acute{r}}_j^{\acute{\acute{c}}_j-1}\big) \prod_{j=1}^{H}\big(k_j^{h_j-1}\big) \prod_{j=1}^{\acute{H}}\big(\acute{k}_j^{\acute{h}_j-1}\big) \prod_{j=1}^{\acute{\acute{H}}}\big(\acute{\acute{k}}_j^{\acute{\acute{H}}_j-1}\big) \Bigg\}.
\end{aligned}$$

*Then*

$$\begin{aligned}
&(1 - x\hat{M})^{-\alpha}(1 - y\hat{N})^{-\beta}(1 - z\hat{R})^{-\delta} \prod\big(t^a, u^e, p^b, \acute{p}^{\acute{b}}, \acute{\acute{p}}^{\acute{\acute{b}}}, q^g, \acute{q}^{\acute{g}}, \acute{\acute{q}}^{\acute{\acute{g}}}, r^c, \acute{r}^{\acute{c}}, \acute{\acute{r}}^{\acute{\acute{c}}}, k^h, \acute{k}^{\acute{h}}, \acute{\acute{k}}^{\acute{\acute{h}}}\big) \\
&\quad = \prod\big(t^a, u^e, p^b, \acute{p}^{\acute{b}}, \acute{\acute{p}}^{\acute{\acute{b}}}, q^g, \acute{q}^{\acute{g}}, \acute{\acute{q}}^{\acute{\acute{g}}}, r^c, \acute{r}^{\acute{c}}, \acute{\acute{r}}^{\acute{\acute{c}}}, k^h, \acute{k}^{\acute{h}}, \acute{\acute{k}}^{\acute{\acute{h}}}\big) \\
&\qquad \times F^{(3)}\left[\begin{matrix} (a_A) :: (b_B); (\acute{b}_{\acute{B}}); (\acute{\acute{b}}_{\acute{\acute{B}}}) : \alpha, (c_C); \beta, (\acute{c}_{\acute{C}}); \delta, (\acute{\acute{c}}_{\acute{\acute{C}}}); \\ (e_E) :: (g_G); (\acute{g}_{\acute{G}}); (\acute{\acute{g}}_{\acute{\acute{G}}}) : \quad (h_H); \quad (\acute{h}_{\acute{H}}); \quad (\acute{\acute{h}}_{\acute{\acute{H}}}); \end{matrix} \; x, y, z\right]
\end{aligned} \tag{1.33}$$

*and*

$$\begin{aligned}
&(1 - x\hat{M} - y\hat{N} - z\hat{R})^{-\alpha} \prod\big(t^a, u^e, p^b, \acute{p}^{\acute{b}}, \acute{\acute{p}}^{\acute{\acute{b}}}, q^g, \acute{q}^{\acute{g}}, \acute{\acute{q}}^{\acute{\acute{g}}}, r^c, \acute{r}^{\acute{c}}, \acute{\acute{r}}^{\acute{\acute{c}}}, k^h, \acute{k}^{\acute{h}}, \acute{\acute{k}}^{\acute{\acute{h}}}\big) \\
&\quad = \prod\big(t^a, u^e, p^b, \acute{p}^{\acute{b}}, \acute{\acute{p}}^{\acute{\acute{b}}}, q^g, \acute{q}^{\acute{g}}, \acute{\acute{q}}^{\acute{\acute{g}}}, r^c, \acute{r}^{\acute{c}}, \acute{\acute{r}}^{\acute{\acute{c}}}, k^h, \acute{k}^{\acute{h}}, \acute{\acute{k}}^{\acute{\acute{h}}}\big) \\
&\qquad \times F^{(3)}\left[\begin{matrix} \alpha, (a_A) :: (b_B); (\acute{b}_{\acute{B}}); (\acute{\acute{b}}_{\acute{\acute{B}}}) : (c_C); (\acute{c}_{\acute{C}}); (\acute{\acute{c}}_{\acute{\acute{C}}}); \\ (e_E) :: (g_G); (\acute{g}_{\acute{G}}); (\acute{\acute{g}}_{\acute{\acute{G}}}) : (h_H); (\acute{h}_{\acute{H}}); (\acute{\acute{h}}_{\acute{\acute{H}}}); \end{matrix} \; x, y, z\right].
\end{aligned} \tag{1.34}$$

**Proof.** The proof is a direct application of the formulas (1.24) and (1.18). □

If in (1.33) and (1.34), we let

$$z = B = \acute{B} = \acute{\acute{B}} = G = \acute{G} = \acute{\acute{G}} = 0$$

then adjusting the parameters, Eqs. (1.33) and (1.34) reduce to operational images for the double hypergeometric series $F^{(2)}[x, y]$ defined by (1.8) as follows:

$$\begin{aligned}
&\big(1 - x\hat{D}[t_{(A)}; u_{(E)}]\hat{D}[p_{(B)}; q_{(G)}]\big)^{-\alpha}\big(1 - y\hat{D}[t_{(A)}; u_{(E)}]\hat{D}[\acute{p}_{(\acute{B})}; \acute{q}_{(\acute{G})}]\big)^{-\beta} \prod\big(t^a, u^e, p^b, q^g, \acute{p}^{\acute{b}}, \acute{q}^{\acute{g}}\big) \\
&\quad = \prod\big(t^a, u^e, p^b, q^g, \acute{p}^{\acute{b}}, \acute{q}^{\acute{g}}\big) F^{A:B+1;\acute{B}+1}_{E:G;\acute{G}}\left[\begin{matrix} (a_A) : \alpha, (b_B); \beta, (\acute{b}_{(\acute{B})}); \\ (e_E) : \quad (g_G); \quad (\acute{g}_{(\acute{G})}) \end{matrix} \; x, y\right],
\end{aligned} \tag{1.35}$$

and

$$\begin{aligned}
&\big(1 - x\hat{D}[t_{(A)}; u_{(E)}]\hat{D}[p_{(B)}; q_{(G)}] - y\hat{D}[t_{(A)}; u_{(E)}]\hat{D}[\acute{p}_{(\acute{B})}; \acute{q}_{(\acute{G})}]\big)^{-\alpha} \prod\big(t^a, u^e, p^b, q^g, \acute{p}^{\acute{b}}, \acute{q}^{\acute{g}}\big) \\
&\quad = \prod\big(t^a, u^e, p^b, q^g, \acute{p}^{\acute{b}}, \acute{q}^{\acute{g}}\big) F^{A+1:B;\acute{B}}_{E:G;\acute{G}}\left[\begin{matrix} \alpha, (a_A) : (b_B); (\acute{b}_{(\acute{B})}); \\ (e_E) : (g_G); (\acute{g}_{(\acute{G})}) \end{matrix} \; x, y\right],
\end{aligned} \tag{1.36}$$

respectively, where, for convenience

$$\begin{aligned}
&\prod\left(t^a, u^e, p^b, q^g, \acute{p}^{\acute{b}}, \acute{q}^{\acute{g}}\right) \\
&\quad = \left\{ \prod_{j=1}^{A}\left(t_j^{a_j-1}\right) \prod_{j=1}^{E}\left(u_j^{e_j-1}\right) \prod_{j=1}^{B}\left(p_j^{b_j-1}\right) \prod_{j=1}^{G}\left(q_j^{g_j-1}\right) \prod_{j=1}^{\acute{B}}\left(\acute{p}_j^{\acute{b}_j-1}\right) \prod_{j=1}^{\acute{G}}\left(\acute{q}_j^{\acute{g}_j-1}\right) \right\}.
\end{aligned}$$

In the next section we will benefit from the operational images introduced in this section for the derivation of certain decomposition formulas involving quadruple series.

## 2. Burchnall–Chaundy type decomposition formulas

Burchnall and Chaundy [1,2] and Chaundy [3] systematically presented a number of expansion and decomposition formulas for Appell double hypergeometric series (1.4)–(1.7) in series of simpler hypergeometric functions. Their method based on certain inverse pairs of symbolic operators. The method of Burchnall and Chaundy is applied mutatis mutandis by a number of workers, for example, Pandey [12], Srivastava [17], Hasanov and Srivastava [7,8], Hasanov et al. [9], Hasanov and Turaev [10], Singhal and Bhati [16] in order to derive expansion formulas involving Lauricella's triple functions $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$ and $F_D^{(3)}$, Srivastava's functions $H_A$, $H_B$ and $H_C$ and Horns' functions $G_1$, $G_2$ and $G_3$. In this section we apply a new method based upon the symbolic operational images (1.25) and (1.33)–(1.36) in order to derive decomposition formulas of Burchnall–Chaundy type. First of all, for notational convenience, we write

$$\Omega(m,n,p,q) = \frac{\prod_{j=1}^{A}(a_j)_{m+n}\prod_{j=1}^{F}(f_j)_{p+q}\prod_{j=1}^{B}(b_j)_m\prod_{j=1}^{\acute{B}}(\acute{b}_j)_n\prod_{j=1}^{C}(c_j)_p\prod_{j=1}^{\acute{C}}(\acute{c}_j)_q}{\prod_{j=1}^{E}(e_j)_{m+n}\prod_{j=1}^{D}(d_j)_{p+q}\prod_{j=1}^{G}(g_j)_m\prod_{j=1}^{\acute{G}}(\acute{g}_j)_n\prod_{j=1}^{H}(h_j)_p\prod_{j=1}^{\acute{H}}(\acute{h}_j)_q}, \tag{2.1}$$

$$\begin{aligned}
\Theta(m,n,p,q) &= \frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\prod_{j=1}^{B}(b_j)_{m+n}\prod_{j=1}^{\acute{B}}(\acute{b}_j)_{n+p}\prod_{j=1}^{\acute{\acute{B}}}(\acute{\acute{b}}_j)_{p+m}\prod_{j=1}^{C}(c_j)_m}{\prod_{j=1}^{E}(e_j)_{m+n+p}\prod_{j=1}^{G}(g_j)_{m+n}\prod_{j=1}^{\acute{G}}(\acute{g}_j)_{n+p}\prod_{j=1}^{\acute{\acute{G}}}(\acute{\acute{g}}_j)_{p+m}\prod_{j=1}^{H}(h_j)_m} \\
&\quad \times \frac{\prod_{j=1}^{\acute{C}}(\acute{c}_j)_n\prod_{j=1}^{\acute{\acute{C}}}(\acute{\acute{c}}_j)_p\prod_{j=1}^{F}(f_j)_q}{\prod_{j=1}^{\acute{H}}(\acute{h}_j)_n\prod_{j=1}^{\acute{\acute{H}}}(\acute{\acute{h}}_j)_p\prod_{j=1}^{D}(d_j)_q}.
\end{aligned} \tag{2.2}$$

By virtue of the operator (1.17) (cf. (1.18)), the operational images (1.25) and (1.33)–(1.36) and by employing the above notation, we find each of the following inverse pairs of the decomposition formulas, where we put each inverse series directly after the corresponding direct series:

$$\begin{aligned}
&\sum_{m,n,p,q=0}^{\infty} \Omega(m,n,p,q)\frac{(\alpha)_{m+p}(\beta)_n(\delta)_q x^m y^n z^p w^q}{m!n!p!q!} \\
&\quad = \sum_{m,n,p,q,s=0}^{\infty} \Omega(m+s,n,p+s,q)\frac{(\alpha)_s(\alpha+s)_m(\beta)_n(\alpha+s)_p(\delta)_q x^m y^n z^p w^q (xz)^s}{s!m!n!p!q!},
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&\sum_{m,n,p,q=0}^{\infty} \Omega(m,n,p,q)\frac{(\alpha)_m(\beta)_n(\alpha)_p(\delta)_q x^m y^n z^p w^q}{m!n!p!q!} \\
&\quad = \sum_{m,n,p,q,s=0}^{\infty} \Omega(m+s,n,p+s,q)\frac{(\alpha)_{m+p+s}(\beta)_n(\delta)_q x^m y^n z^p w^q (-xz)^s}{m!n!p!q!s!},
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
&\sum_{m,n,p,q=0}^{\infty} \Omega(m,n,p,q)\frac{(\alpha)_{m+p}(\beta)_{n+q} x^m y^n z^p w^q}{m!n!p!q!} \\
&\quad = \sum_{m,n,p,q,s,k=0}^{\infty} \Omega(m+s,n+k,p+s,q+k) \\
&\qquad \times \frac{(\alpha)_s(\beta)_k(\alpha+s)_m(\beta+k)_n(\alpha+s)_p(\beta+k)_q x^m y^n z^p w^q (xz)^s (yw)^k}{m!n!p!q!s!k!},
\end{aligned} \tag{2.5}$$

$$\sum_{m,n,p,q=0}^{\infty}\Omega(m,n,p,q)\frac{(\alpha)_m(\beta)_n(\alpha)_p(\beta)_q x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s,k=0}^{\infty}\Omega(m+s,n+k,p+s,q+k)\frac{(\alpha)_{m+p+s}(\beta)_{n+q+k}x^m y^n z^p w^q(-xz)^s(-yw)^k}{m!n!p!q!s!k!},\tag{2.6}$$

$$\sum_{m,n,p,q=0}^{\infty}\Omega(m,n,p,q)\frac{(\alpha)_{m+n+p}(\delta)_q x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s,k,l,r=0}^{\infty}\Omega(m+s+k+r,n+s+l+r,p+k+l+r,q)$$
$$\times\frac{(\alpha)_K(\alpha+K)_m(\alpha+K)_n(\alpha+K)_p(\delta)_q x^m y^n z^p w^q(xy)^s(xz)^k(yz)^l(-xyz)^r}{m!n!p!q!s!k!l!r!}.\tag{2.7}$$

Here and elsewhere in this paper, for convenience ($K := s+k+l+r$)

$$\sum_{m,n,p,q=0}^{\infty}\Omega(m,n,p,q)\frac{(\alpha)_m(\alpha)_n(\alpha)_p(\delta)_q x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s,k,l,r=0}^{\infty}\Omega(m+s+k+r,n+s+l+r,p+k+l+r,q)$$
$$\times\frac{(\alpha)_{K+m+n+p}(\delta)_q x^m y^n z^p w^q(-xy)^s(-xz)^k(-yz)^l(xyz)^r}{m!n!p!q!s!k!l!r!},\tag{2.8}$$

$$\sum_{m,n,p,q=0}^{\infty}\Omega(m,n,p,q)\frac{(\alpha)_{m+n+p+q}x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s,k,l,r,\hat{n},\hat{p},\hat{q},\hat{s},\hat{k},\hat{l},\hat{r}=0}^{\infty}\Omega(M,N,P,Q)\frac{(\alpha)_L(\alpha+L)_m(\alpha+L)_n(\alpha+L)_p(\alpha+L)_q}{m!n!p!q!s!k!l!r!\hat{n}!\hat{p}!\hat{q}!\hat{s}!\hat{k}!\hat{l}!\hat{r}!}$$
$$\times x^m y^n z^p w^q(xy)^s(xz)^k(xw)^l(yz)^r(yw)^{\hat{n}}(zw)^{\hat{p}}(-xyz)^{\hat{q}}(-xyw)^{\hat{s}}(-xzw)^{\hat{k}}(-yzw)^{\hat{l}}(xyzw)^{\hat{r}}\tag{2.9}$$

$$\left(\begin{array}{ll}M := m+s+k+l+\hat{q}+\hat{s}+\hat{k}+\hat{r}, & N := n+s+r+\hat{n}+\hat{q}+\hat{s}+\hat{l}+\hat{r}\\ P := p+k+r+\hat{p}+\hat{q}+\hat{k}+\hat{l}+\hat{r}, & Q := q+l+\hat{n}+\hat{p}+\hat{s}+\hat{k}+\hat{l}+\hat{r}\\ L := s+k+l+r+\hat{n}+\hat{p}+\hat{q}+\hat{s}+\hat{k}+\hat{l}+\hat{r} & \end{array}\right),\tag{2.10}$$

$$\sum_{m,n,p,q=0}^{\infty}\Omega(m,n,p,q)\frac{(\alpha)_m(\alpha)_n(\alpha)_p(\alpha)_q x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s,k,l,r,\hat{n},\hat{p},\hat{q},\hat{s},\hat{k},\hat{l},\hat{r}=0}^{\infty}\Omega(M,N,P,Q)\frac{(\alpha)_L}{m!n!p!q!s!k!l!r!\hat{n}!\hat{p}!\hat{q}!\hat{s}!\hat{k}!\hat{l}!\hat{r}!}x^m y^n z^p w^q$$
$$\times(-xy)^s(-xz)^k(-xw)^l(-yz)^r(-yw)^{\hat{n}}(-zw)^{\hat{p}}(xyz)^{\hat{q}}(xyw)^{\hat{s}}(xzw)^{\hat{k}}(yzw)^{\hat{l}}(-xyzw)^{\hat{r}}\tag{2.11}$$

where $M$, $N$, $P$, $Q$ and $L$ are given by (2.10),

$$\sum_{m,n,p,q=0}^{\infty}\Omega(m,n,p,q)\frac{(\alpha)_{m+p+q}(\beta)_n x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s,k=0}^{\infty}\Omega(m+s+k,n,p+s,q+k)$$
$$\times\frac{(\alpha)_{s+k}(\alpha+s+k)_m(\beta)_n(\alpha+s+k)_{p+q}x^m y^n z^p w^q(xz)^s(xw)^k}{m!n!p!q!s!k!},\tag{2.12}$$

$$\sum_{m,n,p,q=0}^{\infty}\Omega(m,n,p,q)\frac{(\alpha)_m(\beta)_n(\alpha)_{p+q}x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s,k=0}^{\infty}\Omega(m+s+k,n,p+s,q+k)\frac{(\alpha)_{m+p+q+s+k}(\beta)_n x^m y^n z^p w^q(-xz)^s(-xw)^k}{m!n!p!q!s!k!}, \tag{2.13}$$

$$\sum_{m,n,p,q=0}^{\infty}\Omega(m,n,p,q)\frac{(\alpha)_{m+n+p+q}x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s,k,l,r,\hat{n},\hat{p},\hat{q}=0}^{\infty}\Omega(\bar{M},\bar{N},\bar{P},\bar{Q})\frac{(\alpha)_{\bar{R}}(\alpha+\bar{R})_m(\alpha+\bar{R})_n(\alpha+\bar{R})_{p+q}}{m!n!p!q!s!k!l!r!\hat{n}!\hat{p}!\hat{q}!}$$
$$\times x^m y^n z^p w^q(xy)^s(xz)^k(xw)^l(yz)^r(yw)^{\hat{n}}(-xyz)^{\hat{p}}(-xzw)^{\hat{q}} \tag{2.14}$$

$$\begin{pmatrix}\bar{M}:=m+s+k+l+\hat{p}+\hat{q}, & \bar{N}:=n+s+r+\hat{n}+\hat{p}+\hat{q} \\ \bar{P}:=p+k+r+\hat{p}, \quad \bar{Q}:=q+l+\hat{n}+\hat{q}, & \bar{R}:=s+k+l+r+\hat{n}+\hat{p}+\hat{q}\end{pmatrix}, \tag{2.15}$$

$$\sum_{m,n,p,q=0}^{\infty}\Omega(m,n,p,q)\frac{(\alpha)_m(\alpha)_n(\alpha)_{p+q}x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s,k,l,r,\hat{n},\hat{p},\hat{q}=0}^{\infty}\Omega(\bar{M},\bar{N},\bar{P},\bar{Q})$$
$$\times\frac{(\alpha)_{\bar{R}}x^m y^n z^p w^q(-xy)^s(-xz)^k(-xw)^l(-yz)^r(-yw)^{\hat{n}}(xyz)^{\hat{p}}(xzw)^{\hat{q}}}{m!n!p!q!s!k!l!r!\hat{n}!\hat{p}!\hat{q}!} \tag{2.16}$$

where $\bar{M}$, $\bar{N}$, $\bar{P}$, $\bar{Q}$ and $\bar{R}$ are given by (2.15),

$$\sum_{m,n,p,q=0}^{\infty}\Omega(m,n,p,q)\frac{(\alpha)_{m+n+p+q}x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s,k,l,r=0}^{\infty}\Omega(m+s+k,n+l+r,p+s+l,q+k+r)$$
$$\times\frac{(\alpha)_K(\alpha+K)_{m+n}(\alpha+K)_{p+q}x^m y^n z^p w^q(xz)^s(xw)^k(yz)^l(yw)^r}{m!n!p!q!s!k!l!r!}, \tag{2.17}$$

$$\sum_{m,n,p,q=0}^{\infty}\Omega(m,n,p,q)\frac{(\alpha)_{m+n}(\alpha)_{p+q}x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s,k,l,r=0}^{\infty}\Omega(m+s+k,n+l+r,p+s+l,q+k+r)$$
$$\times\frac{(\alpha)_{m+n+p+q+K}x^m y^n z^p w^q(-xz)^s(-xw)^k(-yz)^l(-yw)^r}{m!n!p!q!s!k!l!r!}, \tag{2.18}$$

$$\sum_{m,n,p,q=0}^{\infty}\Theta(m,n,p,q)\frac{(\alpha)_{m+q}(\beta)_n(\delta)_p x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s=0}^{\infty}\Theta(m+s,n,p,q+s)\frac{(\alpha)_s(\alpha+s)_m(\beta)_n(\delta)_p(\alpha+s)_q x^m y^n z^p w^q(xz)^s}{s!m!n!p!q!}, \tag{2.19}$$

$$\sum_{m,n,p,q=0}^{\infty}\Theta(m,n,p,q)\frac{(\alpha)_m(\beta)_n(\alpha)_p(\delta)_q x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s=0}^{\infty}\Theta(m+s,n,p,q+s)\frac{(\alpha)_{m+q+s}(\beta)_n(\delta)_p x^m y^n z^p w^q(-xz)^s}{m!n!p!q!s!}, \tag{2.20}$$

$$\sum_{m,n,p,q=0}^{\infty}\Theta(m,n,p,q)\frac{(\alpha)_{m+n+q}(\beta)_q x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s,k,l,r=0}^{\infty}\Theta(m+s+k+r,n+s+l+r,p,q+k+l+r)$$
$$\times\frac{(\alpha)_K(\alpha+K)_m(\alpha+K)_n(\alpha+K)_q(\beta)_p x^m y^n z^p w^q (xy)^s (xw)^k (yw)^l (-xyw)^r}{m!n!p!q!s!k!l!r!}, \tag{2.21}$$

$$\sum_{m,n,p,q=0}^{\infty}\Theta(m,n,p,q)\frac{(\alpha)_m(\alpha)_n(\beta)_p(\alpha)_q x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s,k,l,r=0}^{\infty}\Theta(m+s+k+r,n+s+l+r,p,q+k+l+r)$$
$$\times\frac{(\alpha)_{s+k+l+r+m+n+q}(\beta)_p x^m y^n z^p w^q (-xy)^s (-xw)^k (-yw)^l (xyw)^r}{m!n!p!q!s!k!l!r!}, \tag{2.22}$$

$$\sum_{m,n,p,q=0}^{\infty}\Theta(m,n,p,q)\frac{(\alpha)_{m+n+p+q} x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s,k,l,r,\hat{n},\hat{p},\hat{q},\hat{s},\hat{k},\hat{l},\hat{r}=0}^{\infty}\Theta(M,N,P,Q)\frac{(\alpha)_L(\alpha+L)_m(\alpha+L)_n(\alpha+L)_p(\alpha+L)_q}{m!n!p!q!s!k!l!r!\hat{n}!\hat{p}!\hat{q}!\hat{s}!\hat{k}!\hat{l}!\hat{r}!}$$
$$\times x^m y^n z^p w^q (xy)^s (xz)^k (xw)^l (yz)^r (yw)^{\hat{n}} (zw)^{\hat{p}} (-xyz)^{\hat{q}} (-xyw)^{\hat{s}} (-xzw)^{\hat{k}} (-yzw)^{\hat{l}} (xyzw)^{\hat{r}} \tag{2.23}$$

where $M, N, P, Q$ and $L$ are given by (2.10),

$$\sum_{m,n,p,q=0}^{\infty}\Theta(m,n,p,q)\frac{(\alpha)_m(\alpha)_n(\alpha)_p(\alpha)_q x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s,k,l,r,\hat{n},\hat{p},\hat{q},\hat{s},\hat{k},\hat{l},\hat{r}=0}^{\infty}\Theta(M,N,P,Q)\frac{(\alpha)_L}{m!n!p!q!s!k!l!r!\hat{n}!\hat{p}!\hat{q}!\hat{s}!\hat{k}!\hat{l}!\hat{r}!}$$
$$\times x^m y^n z^p w^q (-xy)^s (-xz)^k (-xw)^l (-yz)^r (-yw)^{\hat{n}} (-zw)^{\hat{p}} (xyz)^{\hat{q}} (xyw)^{\hat{s}} (xzw)^{\hat{k}} (yzw)^{\hat{l}} (-xyzw)^{\hat{r}} \tag{2.24}$$

where $M, N, P, Q$ and $L$ are given by (2.10),

$$\sum_{m,n,p,q=0}^{\infty}\Theta(m,n,p,q)\frac{(\alpha)_{m+n+p+q}(\alpha)_q x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s,k,l=0}^{\infty}\Theta(m+s,n+k,p+l,q+s+k+l)$$
$$\times\frac{(\alpha)_{s+k+l}(\alpha+s+k+l)_{m+n+p}(\alpha+s+k+l)_q x^m y^n z^p w^q (xw)^s (yw)^k (zw)^l}{m!n!p!q!s!k!l!}, \tag{2.25}$$

$$\sum_{m,n,p,q=0}^{\infty}\Theta(m,n,p,q)\frac{(\alpha)_{m+n+p}(\alpha)_q x^m y^n z^p w^q}{m!n!p!q!}$$
$$=\sum_{m,n,p,q,s,k,l=0}^{\infty}\Theta(m+s,n+k,p+l,q+s+k+l)$$
$$\times\frac{(\alpha)_{m+n+p+q+s+k+l} x^m y^n z^p w^q (-xw)^s (-yw)^k (-zw)^l}{m!n!p!q!s!k!}, \tag{2.26}$$

provided that each member of the above decomposition formulas exists.

### 2.1. Derivations of the decomposition formulas

Denote, for convenience, the left-hand side of the formula (2.3) by $\Im(x,y,z,w)$ and let

$$\prod\left(t^a,u^e,l^f,v^d,p^b,s^g,r^c,k^h,\acute{p}^{\acute{b}},\acute{s}^{\acute{g}},\acute{r}^{\acute{c}},\acute{k}^{\acute{h}}\right)$$
$$=\left\{\prod_{j=1}^{A}\left(t_j^{a_j-1}\right)\prod_{j=1}^{E}\left(u_j^{e_j-1}\right)\prod_{j=1}^{F}\left(l_j^{f_j-1}\right)\prod_{j=1}^{D}\left(v_j^{d_j-1}\right)\prod_{j=1}^{B}\left(p_j^{b_j-1}\right)\prod_{j=1}^{G}\left(s_j^{g_j-1}\right)\right.$$
$$\left.\times\prod_{j=1}^{C}\left(r_j^{c_j-1}\right)\prod_{j=1}^{H}\left(k_j^{h_j-1}\right)\prod_{j=1}^{\acute{B}}\left(\acute{p}_j^{\acute{b}_j-1}\right)\prod_{j=1}^{\acute{G}}\left(\acute{s}_j^{\acute{g}_j-1}\right)\prod_{j=1}^{\acute{C}}\left(\acute{r}_j^{\acute{c}_j-1}\right)\prod_{j=1}^{\acute{H}}\left(\acute{k}_j^{\acute{h}_j-1}\right)\right\}.$$

Then, in view of (2.1), the relation (1.18) and the symbolic operational image (1.35), we can write

$$\prod\left(t^a,u^e,l^f,v^d,p^b,s^g,r^c,k^h,\acute{p}^{\acute{b}},\acute{s}^{\acute{g}},\acute{r}^{\acute{c}},\acute{k}^{\acute{h}}\right)\Im(x,y,z,w)$$
$$=\sum_{n,q=0}^{\infty}\frac{(\beta)_n(\delta)_q y^n w^q}{n!q!}\hat{D}^n[t_{(A)};u_{(E)}]\hat{D}^n[\acute{p}_{(\acute{B})};\acute{s}_{(\acute{G})}]\hat{D}^q[l_{(F)};v_{(D)}]\hat{D}^q[\acute{r}_{(\acute{C})};\acute{k}_{(\acute{H})}]$$
$$\times\left[1-x\hat{D}[t_{(A)};u_{(E)}]\hat{D}[p_{(B)};s_{(G)}]-z\hat{D}[l_{(F)};v_{(D)}]\hat{D}[r_{(C)};k_{(H)}]\right]^{-\alpha}$$
$$\times\prod\left(t^a,u^e,l^f,v^d,p^b,s^g,r^c,k^h,\acute{p}^{\acute{b}},\acute{s}^{\acute{g}},\acute{r}^{\acute{c}},\acute{k}^{\acute{h}}\right)$$
$$=\sum_{n,q=0}^{\infty}\frac{(\beta)_n(\delta)_q y^n w^q}{n!q!}\hat{D}^n[t_{(A)};u_{(E)}]\hat{D}^n[\acute{p}_{(\acute{B})};\acute{s}_{(\acute{G})}]\hat{D}^q[l_{(F)};v_{(D)}]\hat{D}^q[\acute{r}_{(\acute{C})};\acute{k}_{(\acute{H})}]$$
$$\times\left[\left(1-x\hat{D}[t_{(A)};u_{(E)}]\hat{D}[p_{(B)};s_{(G)}]\right)\left(1-z\hat{D}[l_{(F)};v_{(D)}]\hat{D}[r_{(C)};k_{(H)}]\right)\right.$$
$$\left.-xz\hat{D}[t_{(A)};u_{(E)}]\hat{D}[p_{(B)};s_{(G)}]\hat{D}[l_{(F)};v_{(D)}]\hat{D}[r_{(C)};k_{(H)}]\right]^{-\alpha}$$
$$\times\prod\left(t^a,u^e,l^f,v^d,p^b,s^g,r^c,k^h,\acute{p}^{\acute{b}},\acute{s}^{\acute{g}},\acute{r}^{\acute{c}},\acute{k}^{\acute{h}}\right). \tag{2.27}$$

Now, on making use of (1.24), applying the operator (1.18) and interpreting the resulting expression by means of (2.1), we arrive immediately at the right-hand side of (2.3). To prove the inversion formula (2.4) of (2.3), we start from the right-hand side of formula (2.3) with $s=0$, which is the left-hand side of (2.3). Denote the right-hand side of assertion (2.4) by $\Im^{-1}(x,y,z,w)$. Then, in view of (2.1), the operational image (1.33) and the relation (1.22), we can write

$$\prod\left(t^a,u^e,l^f,v^d,p^b,r^c,k^h,\acute{p}^{\acute{b}},\acute{s}^{\acute{g}},\acute{r}^{\acute{c}},\acute{k}^{\acute{h}}\right)\Im^{-1}(x,y,z,w)$$
$$=\sum_{n,q=0}^{\infty}\frac{(\beta)_n(\delta)_q y^n w^k}{n!q!}\hat{D}^n[t_{(A)};u_{(E)}]\hat{D}^n[\acute{p}_{(\acute{B})};\acute{s}_{(\acute{G})}]\hat{D}^q[l_{(F)};v_{(D)}]\hat{D}^q[\acute{r}_{(\acute{C})};\acute{k}_{(\acute{H})}]$$
$$\times\left(1-x\hat{D}[t_{(A)};u_{(E)}]\hat{D}[p_{(B)};s_{(G)}]-z\hat{D}[l_{(F)};v_{(D)}]\hat{D}[r_{(C)};k_{(H)}]\right.$$
$$\left.+xz\hat{D}[t_{(A)};u_{(E)}]\hat{D}[p_{(B)};s_{(G)}]\hat{D}[l_{(F)};v_{(D)}]\hat{D}[r_{(C)};k_{(H)}]\right)^{-\alpha}$$
$$\times\prod\left(t^a,u^e,l^f,v^d,p^b,r^c,k^h,\acute{p}^{\acute{b}},\acute{s}^{\acute{g}},\acute{r}^{\acute{c}},\acute{k}^{\acute{h}}\right). \tag{2.28}$$

Now, starting from (2.28) and making use of (1.24) and (1.18) and interpreting the resulting expression by means of (2.1), we get the right-hand side of (2.4). This completes the proof of the inverse pairs (2.3) and (2.4) under the assumption that each member of (2.3) and (2.4) exists. The proof of each of the inverse pairs {(2.5), (2.6)}, {(2.7), (2.8)}, {(2.9), (2.11)}, {(2.12), (2.13)}, {(2.14), (2.16)}, {(2.17), (2.18)}, {(2.19), (2.20)}, {(2.21), (2.22)}, {(2.23), (2.24)} and {(2.25), (2.26)} runs parallel to that of the inverse pair {(2.3), (2.4)} then we skip the details.

## 3. Applications

By assigning suitable special values to the coefficients of the quadruple series $\Omega(m,n,p,q)$ and $\Theta(m,n,p,q)$, inverse pairs decomposition formulas of the previous section can indeed be specialized to establish a large number of (new and known) decomposition formulas including those given by Burchnall and Chaundy [1,2]. In this section we choose to mention only some direct series of our main results (2.3), (2.5), (2.7), (2.9), (2.12), (2.14), (2.17), (2.19), (2.21), (2.23) and (2.25). First, by appropriately choosing the quadruple series $\Omega(m,n,p,q)$ and $\Theta(m,n,p,q)$ in accordance with the definitions of the quadruple series of Exton [6] $K_2$, $K_5$, $K_8$, $K_9$, $K_{10}$, $K_{11}$, $K_{12}$ and $K_{13}$ and of Sharma and Parihar [14] $F_4^{(4)}$, $F_8^{(4)}$, $F_{21}^{(4)}$, $F_{22}^{(4)}$, $F_{45}^{(4)}$, $F_{49}^{(4)}$ and $F_{57}^{(4)}$ we shall obtain the following decomposition formulas:

$$\begin{aligned}
&F_{49}^{(4)}[a,a,f,f,\alpha,\beta,\alpha,\delta;e,e,d,d;x,y,z,w]\\
&\quad=\sum_{s=0}^{\infty}\frac{(a)_s(f)_s(\alpha)_s(xz)^s}{(e)_s(d)_s s!}\\
&\qquad\times F_1[a+s,\alpha+s,\beta;e;x,y]F_1[f+s,\alpha+s,\delta;d;z,w],
\end{aligned}\tag{3.1}$$

$$\begin{aligned}
&F_{8}^{(4)}[a,a,f,f,\alpha,\beta,\alpha,\delta;g,\acute{g},h,\acute{h};x,z,w,y]\\
&\quad=\sum_{s=0}^{\infty}\frac{(a)_s(f)_s(\alpha)_s(xz)^s}{(g)_s(h)_s s!}\\
&\qquad\times F_2[a+s,\alpha+s,\beta;g+s,\acute{g};x,y]F_2[f+s,\alpha+s,\delta,;h+s,\acute{h};z,w],
\end{aligned}\tag{3.2}$$

$$\begin{aligned}
&F_{57}^{(4)}[a,a,f,f,\alpha,\beta,\alpha,\beta,;e,e,d,d;x,y,z,w]\\
&\quad=\sum_{s,k=0}^{\infty}\frac{(a)_{s+k}(f)_{s+k}(\alpha)_s(\beta)_k(xz)^s(yw)^k}{(e)_{s+k}(d)_{s+k}s!k!}\\
&\qquad\times F_1[a+s+k,\alpha+s,\beta+k;e+s+k;x,y]F_1[f+s+k,\alpha+s,\beta+k;d+s+k;z,w],
\end{aligned}\tag{3.3}$$

$$\begin{aligned}
&F_{46}^{(4)}[\alpha,\alpha,\alpha,\delta,b,\acute{b},c,\acute{c};e,e,d,d;x,y,z,w]\\
&\quad=\sum_{s,k,l,r=0}^{\infty}\frac{(\alpha)_K(b)_{s+k+r}(\acute{b})_{s+l+r}(c)_{k+l+r}(xy)^s(xz)^k(yz)^l(-xyz)^r}{(e)_{K+s+r}(d)_{k+l+r}s!k!l!r!}\\
&\qquad\times F_3[\alpha+K,\alpha+K,b+s+k+r,\acute{b}+s+l+r;e+K+s+r;x,y]\\
&\qquad\times F_3[\alpha+K,\delta,c+k+l+r,\acute{c};d+k+l+r;z,w],
\end{aligned}\tag{3.4}$$

$$\begin{aligned}
&K_{13}[\alpha,\alpha,\alpha,\alpha,b,\acute{b},c,\acute{c};e,e,h,\acute{h};x,y,z,w]\\
&\quad=\sum_{s,k,l,r,\hat{n},\hat{p},\hat{q},\hat{s},\hat{k},\hat{l},\hat{r}=0}^{\infty}\frac{(\alpha)_{\tilde{L}}(b)_{\tilde{M}}(\acute{b})_{\tilde{N}}(c)_{\tilde{P}}(\acute{c})_{\tilde{Q}}}{(e)_{\tilde{M}+\tilde{N}}(h)_{\tilde{P}}(\acute{h})_{\tilde{Q}}s!k!l!r!\hat{n}!\hat{p}!\hat{q}!\hat{s}!\hat{k}!\hat{l}!\hat{r}!}\\
&\qquad\times(xy)^s(xz)^k(xw)^l(yz)^r(yw)^{\hat{n}}(zw)^{\hat{p}}(-xyz)^{\hat{q}}(-xyw)^{\hat{s}}(-xzw)^{\hat{k}}(-yzw)^{\hat{l}}(xyzw)^{\hat{r}}\\
&\qquad\times F_3[\alpha+\tilde{L},\alpha+\tilde{L},b+\tilde{M},\acute{b}+\tilde{N};e+\tilde{M}+\tilde{N};x,y]\\
&\qquad\times {}_2F_1[\alpha+\tilde{L};h+\tilde{P},\acute{h}+\tilde{P};z]{}_2F_1[\alpha+\tilde{L},\acute{c}+\tilde{Q};\acute{h}+\tilde{Q};w]
\end{aligned}\tag{3.5}$$

here and elsewhere in this paper, for convenience

$$\begin{pmatrix}
\tilde{M}:=s+k+l+\hat{q}+\hat{s}+\hat{k}+\hat{r}, & \tilde{N}:=s+r+\hat{n}+\hat{q}+\hat{s}+\hat{l}+\hat{r}\\
\tilde{P}:=k+r+\hat{p}+\hat{q}+\hat{k}+\hat{l}+\hat{r}, & \tilde{Q}:=l+\hat{n}+\hat{p}+\hat{s}+\hat{k}+\hat{l}+\hat{r}\\
\tilde{L}:=s+k+l+r+\hat{n}+\hat{p}+\hat{q}+\hat{s}+\hat{k}+\hat{l}+\hat{r} &
\end{pmatrix},\tag{3.6}$$

$$\begin{aligned}
&F_{45}^{(4)}[\alpha,\alpha,\alpha,\beta,a,c,\acute{c},a;e,d,d,e;x,z,w,y]\\
&\quad=\sum_{s,k=0}^{\infty}\frac{(a)_{s+k}(\alpha)_{s+k}(c)_s(\acute{c})_k(xz)^s(xw)^k}{(e)_{s+k}(d)_{s+k}s!k!}\\
&\qquad\times F_1[a+s+k,\alpha+s+k,\beta;e+s+k;x,y]F_1[\alpha+s+k,c+s,\acute{c}+k;d+s+k;z,w],
\end{aligned}\tag{3.7}$$

$$\begin{aligned}
&F_{21}^{(4)}[\alpha,\alpha,\alpha,\beta,f,f,b,\acute{b};h,\acute{h},e,e;z,w,x,y]\\
&\quad=\sum_{s,k=0}^{\infty}\frac{(\alpha)_{s+k}(b)_{s+k}(f)_{s+k}(xz)^s(xw)^k}{(e)_{s+k}(h)_s(\acute{h})_k s!k!}\\
&\qquad\times F_3[\alpha+s+k,\beta,b+s+k,\acute{b};e+s+k;x,y]F_4[\alpha+s+k,f+s+k,;h+s,\acute{h}+k;z,w],
\end{aligned}\tag{3.8}$$

$$K_{10}[\alpha,\alpha,\alpha,\alpha,a,a,c,\acute{c};g,\acute{g},h,\acute{h};x,z,w,y]$$
$$=\sum_{s,k,l,r,\hat{n},\hat{p},\hat{q}=0}^{\infty}\frac{(\alpha)_{\bar{\bar{R}}}(a)_{\bar{\bar{M}}+\bar{\bar{N}}}(c)_{\bar{\bar{P}}}(\acute{c})_{\bar{\bar{Q}}}}{(g)_{\bar{\bar{M}}}(\acute{g})_{\bar{\bar{N}}}(h)_{\bar{\bar{P}}}(\acute{h})_{\bar{\bar{Q}}}\,s!k!l!r!\hat{n}!\hat{p}!\hat{q}!}(xy)^s(xz)^k(xw)^l(yz)^r(yw)^{\hat{n}}(-xyz)^{\hat{p}}(-xzw)^{\hat{q}}$$
$$\times F_2[a+\bar{\bar{M}}+\bar{\bar{N}},\alpha+\bar{\bar{R}},\alpha+\bar{\bar{R}};g+\bar{\bar{M}},\acute{g}+\bar{\bar{N}};x,y]F_2[\alpha+\bar{\bar{R}},c+\bar{\bar{P}},\acute{c}+\bar{\bar{Q}};h+\bar{\bar{P}},\acute{h}+\bar{\bar{Q}};z,w] \tag{3.9}$$
$$\begin{pmatrix}\bar{\bar{M}}:=s+k+l+\hat{p}+\hat{q}, \quad \bar{\bar{N}}:=s+r+\hat{n}+\hat{p}+\hat{q}\\ \bar{\bar{P}}:=k+r+\hat{p}, \quad \bar{\bar{Q}}:=l+\hat{n}+\hat{q}\\ \bar{\bar{R}}:=s+k+l+r+\hat{n}+\hat{p}+\hat{q}\end{pmatrix}, \tag{3.10}$$
$$K_{12}[\alpha,\alpha,\alpha,\alpha,b,\acute{b},c,\acute{c};e,e,d,d;x,y,z,w]$$
$$=\sum_{s,k,l,r,\hat{n},\hat{p},\hat{q}=0}^{\infty}\frac{(\alpha)_{\bar{\bar{R}}}(b)_{\bar{\bar{M}}}(\acute{b})_{\bar{\bar{N}}}(c)_{\bar{\bar{P}}}(\acute{c})_{\bar{\bar{Q}}}}{(e)_{\bar{\bar{M}}+\bar{\bar{N}}}(d)_{\bar{\bar{P}}+\bar{\bar{Q}}}\,s!k!l!r!\hat{n}!\hat{p}!\hat{q}!}(xy)^s(xz)^k(xw)^l(yz)^r(yw)^{\hat{n}}(-xyz)^{\hat{p}}(-xzw)^{\hat{q}}$$
$$\times F_3[\alpha+\bar{\bar{R}},\alpha+\bar{\bar{R}},b+\bar{\bar{M}},\acute{b}+\bar{\bar{N}};e+\bar{\bar{M}}+\bar{\bar{N}};x,y]F_1[\alpha+\bar{\bar{R}},c+\bar{\bar{P}},\acute{c}+\bar{\bar{Q}};d+\bar{\bar{P}}+\bar{\bar{Q}};z,w] \tag{3.11}$$

where $\bar{\bar{M}}$, $\bar{\bar{N}}$, $\bar{\bar{P}}$, $\bar{\bar{Q}}$ and $\bar{\bar{R}}$ are given, as before, by (3.10),

$$K_9[\alpha,\alpha,\alpha,\alpha,a,c,\acute{c};e,h,\acute{h};x,y,y,w]$$
$$=\sum_{s,k,l,r,\hat{n},\hat{p},\hat{q}=0}^{\infty}\frac{(\alpha)_{\bar{\bar{R}}}(a)_{\bar{\bar{M}}+\bar{\bar{N}}}(c)_{\bar{\bar{P}}}(\acute{c})_{\bar{\bar{Q}}}}{(e)_{\bar{\bar{M}}+\bar{\bar{N}}}(h)_{\bar{\bar{P}}}(\acute{h})_{\bar{\bar{Q}}}\,s!k!l!r!\hat{n}!\hat{p}!\hat{q}!}(xy)^s(xz)^k(xw)^l(yz)^r(yw)^{\hat{n}}(-xyz)^{\hat{p}}(-xzw)^{\hat{q}}$$
$$\times F_1[a+\bar{\bar{M}}+\bar{\bar{N}},\alpha+\bar{\bar{R}},\alpha;e+\bar{\bar{M}}+\bar{\bar{N}};x,y]F_2[\alpha+\bar{\bar{R}},c+\bar{\bar{P}},\acute{c}+\bar{\bar{P}};h+\bar{\bar{P}},\acute{h}+\bar{\bar{Q}};z,w] \tag{3.12}$$

where $\bar{\bar{M}}$, $\bar{\bar{N}}$, $\bar{\bar{P}}$, $\bar{\bar{Q}}$ and $\bar{\bar{R}}$ are given, as before, by (3.10),

$$K_5[\alpha,\alpha,\alpha,\alpha,a,a,f,f;g,\acute{g},h,\acute{h};x,y,z,w]$$
$$=\sum_{s,k,l,r=0}^{\infty}\frac{(\alpha)_K(a)_K(f)_K(xz)^s(xw)^k(yz)^l(yw)^r}{(g)_{s+k}(\acute{g})_{l+r}(h)_{s+l}(\acute{h})_{k+r}s!k!l!r!}$$
$$\times F_4[a+K,\alpha+K;g+s+k,\acute{g}+l+r;x,y]F_4[f+K,\alpha+K;h+s+l,\acute{h}+k+r;z,w], \tag{3.13}$$
$$F_{22}^{(4)}[a,a,a,f,\alpha,\beta,\delta,\alpha;g,g,\acute{\acute{c}},d;x,y,z,w]$$
$$=\sum_{s=0}^{\infty}\frac{(\alpha)_s(a)_s(f)_s(xz)^s}{(g)_s(\acute{c})_s(d)_s s!}{}_2F_1[\alpha+s;,f+s;d+s;w]$$
$$\times F_G[a+s,a+s,a+s,\delta,\alpha+s,\beta;\acute{\acute{c}},g+s,g+s;z,x,y], \tag{3.14}$$
$$F_4^{(4)}[a,a,a,f,\alpha,\alpha,\acute{\acute{c}},\alpha;h,\acute{h},\acute{\acute{h}},d;x,y,z,w]$$
$$=\sum_{s,k,l,r=0}^{\infty}\frac{(\alpha)_K(a)_{K+s+r}(f)_{k+l+r}(xy)^s(xw)^k(yw)^l(-xyw)^r}{(h)_{s+l+r}(\acute{h})_{s+l+r}(d)_{k+l+r}s!k!l!r!}$$
$$\times {}_2F_1[\alpha+K;,f+k+l+r;d+k+l+r;w]$$
$$\times F_A^{(3)}[\lambda+K+s+r,\alpha+K,\alpha+K,\acute{\acute{c}};h+s+k+r,\acute{h}+s+l+r,\acute{\acute{h}};x,y,z], \tag{3.15}$$
$$K_{13}[\alpha,\alpha,\alpha,\alpha,c,\acute{c},\acute{\acute{c}},f;g,g,\acute{h},d;x,y,z,w]$$
$$=\sum_{s,k,l,r,\hat{n},\hat{p},\hat{q},\hat{s},\hat{k},\hat{l},\hat{r}=0}^{\infty}\frac{(\alpha)_{\tilde{L}}(c)_{\tilde{M}}(\acute{c})_{\tilde{N}}(\acute{\acute{c}})_{\tilde{P}}(f)_{\tilde{Q}}}{(g)_{\tilde{M}+\tilde{N}}(\acute{h})_{\tilde{P}}(d)_{\tilde{Q}}\,s!k!l!r!\hat{n}!\hat{p}!\hat{q}!\hat{s}!\hat{k}!\hat{l}!\hat{r}!}$$
$$\times (xy)^s(xz)^k(xw)^l(yz)^r(yw)^{\hat{n}}(zw)^{\hat{p}}(-xyz)^{\hat{q}}(-xyw)^{\hat{s}}(-xzw)^{\hat{k}}(-yzw)^{\hat{l}}(xyzw)^{\hat{r}}$$
$$\times F_3[\alpha+\tilde{L},\alpha+\tilde{L},c+\tilde{M},\acute{c}+\tilde{N};g+\tilde{M}+\tilde{N};x,y]$$
$$\times {}_2F_1[\alpha+\tilde{L},\acute{\acute{c}}+\tilde{P};\acute{h}+\tilde{P};z]{}_2F_1[\alpha+\tilde{L},f+\tilde{Q};d+\tilde{Q};w], \tag{3.16}$$

$$K_2[\alpha,\alpha,\alpha,\alpha,a,a,a,f;c,\acute{c},\acute{\acute{c}},d;x,y,z,w]$$
$$=\sum_{s,k,l,r,\hat{n},\hat{p},\hat{q},\hat{s},\hat{k},\hat{l},\hat{r}=0}^{\infty}\frac{(\alpha)_{\tilde{L}}(a)_{\tilde{M}+\tilde{N}+\tilde{P}}(f)_{\tilde{Q}}}{(h)_{\tilde{M}}\acute{h}_{\tilde{N}}(\acute{\acute{h}})_{\tilde{P}}(d)_{\tilde{Q}}\,s!k!l!r!\hat{n}!\hat{p}!\hat{q}!\hat{s}!\hat{k}!\hat{l}!\hat{r}!}$$
$$\times (xy)^s(xz)^k(xw)^l(yz)^r(yw)^{\hat{n}}(zw)^{\hat{p}}(-xyz)^{\hat{q}}(-xyw)^{\hat{s}}(-xzw)^{\hat{k}}(-yzw)^{\hat{l}}(xyzw)^{\hat{r}}$$
$$\times F_A^{(3)}[a++\tilde{M}+\tilde{N}+\tilde{P},\alpha+\tilde{L},\alpha+\tilde{L},\alpha+\tilde{L};h+\tilde{M},\acute{h}+\tilde{N},\acute{\acute{h}}+\tilde{N};x,y,z]$$
$$\times {}_2F_1[\alpha+\tilde{L},f+\tilde{Q};d+\tilde{Q};z], \tag{3.17}$$

$$K_{10}[\alpha,\alpha,\alpha,\alpha,b,b,\acute{\acute{c}},f;h,\acute{h},\acute{\acute{h}},d;x,y,z,w]$$
$$=\sum_{s,k,l,r,\hat{n},\hat{p},\hat{q},\hat{s},\hat{k},\hat{l},\hat{r}=0}^{\infty}\frac{(\alpha)_{\tilde{L}}(b)_{\tilde{M}+\tilde{N}}(\acute{\acute{c}})_{\tilde{P}}(f)_{\tilde{Q}}}{(h)_{\tilde{M}}(\acute{h})_{\tilde{N}}(\acute{\acute{h}})_{\tilde{P}}(d)_{\tilde{Q}}\,s!k!l!r!\hat{n}!\hat{p}!\hat{q}!\hat{s}!\hat{k}!\hat{l}!\hat{r}!}$$
$$\times (xy)^s(xz)^k(xw)^l(yz)^r(yw)^{\hat{n}}(zw)^{\hat{p}}(-xyz)^{\hat{q}}(-xyw)^{\hat{s}}(-xzw)^{\hat{k}}(-yzw)^{\hat{l}}(xyzw)^{\hat{r}}$$
$$\times F_2[b+\acute{M}+\acute{N},\alpha+\tilde{L},\alpha+\tilde{L};h+\tilde{M},\acute{h}+\tilde{N};x,y]$$
$$\times {}_2F_1[\alpha+\tilde{L},\acute{\acute{c}}+\tilde{P};\acute{\acute{h}}+\tilde{P};z]\,{}_2F_1[\alpha+\tilde{L},f+\tilde{Q};d+\tilde{Q};w], \tag{3.18}$$

$$K_{11}[\alpha,\alpha,\alpha,\alpha,c,\acute{c},\acute{\acute{c}},f;e,e,e,d;x,y,z,w]$$
$$=\sum_{s,k,l,r,\hat{n},\hat{p},\hat{q},\hat{s},\hat{k},\hat{l},\hat{r}=0}^{\infty}\frac{(\alpha)_{\tilde{L}}(c)_{\tilde{M}}(\acute{c})_{\tilde{N}}(\acute{\acute{c}})_{\tilde{P}}(f)_{\tilde{Q}}}{(e)_{\tilde{M}+\tilde{N}}(d)_{\tilde{Q}}\,s!k!l!r!\hat{n}!\hat{p}!\hat{q}!\hat{s}!\hat{k}!\hat{l}!\hat{r}!}$$
$$\times (xy)^s(xz)^k(xw)^l(yz)^r(yw)^{\hat{n}}(zw)^{\hat{p}}(-xyz)^{\hat{q}}(-xyw)^{\hat{s}}(-xzw)^{\hat{k}}(-yzw)^{\hat{l}}(xyzw)^{\hat{r}}$$
$$\times F_B^{(3)}[\alpha+\tilde{L},\alpha+\tilde{L},\alpha+\tilde{L},c+\tilde{M},\acute{c}+\tilde{N},\acute{\acute{c}}+\tilde{P};e+\tilde{M}+\tilde{N};x,y,z]$$
$$\times {}_2F_1[\alpha+\tilde{L},f+\tilde{Q};d+\tilde{Q};w], \tag{3.19}$$

$$K_2[\alpha,\alpha,\alpha,\alpha,a,a,a,f;h,\acute{h},\acute{\acute{h}},d;x,y,z,w]$$
$$=\sum_{s,k,l=0}^{\infty}\frac{(\alpha)_{s+k+l}(a)_{s+k+l}(f)_{s+k+l}(xw)^s(yw)^k(zw)^l}{(h)_s(\acute{h})_k(\acute{\acute{h}})_l(d)_{s+k+l}s!k!l!}$$
$$\times F_C^{(3)}[\alpha+s+k+l,a+s+k+l;h+s,\acute{h}+k,\acute{\acute{h}}+l;x,y,z]$$
$$\times {}_2F_1[\alpha+s+k+l,f+s+k+l;d+s+k+l;w], \tag{3.20}$$

$$K_{10}[\alpha,\alpha,\alpha,\alpha,b,b,\acute{\acute{c}},f;h,\acute{h},\acute{\acute{h}},d;x,y,z,w]$$
$$=\sum_{s,k,l=0}^{\infty}\frac{(\alpha)_{s+k+l}(f)_{s+k+l}(b)_{s+k}(\acute{\acute{c}})_l(xw)^s(yw)^k(zw)^l}{(h)_s(\acute{h})_k(\acute{\acute{h}})_l(d)_{s+k+l}s!k!l!}$$
$$\times F_E[\alpha+s+k+l,\alpha+s+k+l,\alpha+s+k+l,\acute{\acute{c}}+l,b+s+k,b+s+k;\acute{\acute{h}}+l,h+s,\acute{h}+k;z,x,y]$$
$$\times {}_2F_1[\alpha+s+k+l,f+s+k+l;d+s+k+l;w], \tag{3.21}$$

$$K_8[\alpha,\alpha,\alpha,\alpha,b,b,\acute{\acute{c}},f;\acute{\acute{g}},\acute{c},\acute{\acute{g}},d;x,y,z,w]$$
$$=\sum_{s,k,l=0}^{\infty}\frac{(\alpha)_{s+k+l}(f)_{s+k+l}(b)_{s+k}(\acute{\acute{c}})_l(xw)^s(yw)^k(zw)^l}{(\acute{\acute{g}})_{s+l}(h)_s(\acute{c})_k(d)_{s+k+l}s!k!l!}$$
$$\times F_F[\alpha+s+k+l,\alpha+s+k+l,\alpha+s+k+l,b+s+k,\acute{\acute{c}}+l,b+s+k;\acute{c}+k,\acute{\acute{g}}+s+l,\acute{\acute{g}}+s+l;y,z,x]$$
$$\times {}_2F_1[\alpha+s+k+l,f+s+k+l;d+s+k+l;w], \tag{3.22}$$

$$K_{13}[\alpha,\alpha,\alpha,\alpha,c,\acute{c},\acute{\acute{c}},f;b,b,\acute{\acute{h}},d;x,y,z,w]$$
$$=\sum_{s,k,l=0}^{\infty}\frac{(\alpha)_{s+k+l}(f)_{s+k+l}(c)_s(\acute{c})_k(\acute{\acute{c}})_l(xw)^s(yw)^k(zw)^l}{(b)_{s+k}(\acute{\acute{h}})_l(d)_{s+k+l}s!k!l!}$$

$$\times F_G[\alpha+s+k+l,\alpha+s+k+l,\alpha+s+k+l,\acute{\acute{c}}+l,c+s,\acute{c}+k;\acute{\acute{h}}+l,b+s+k,b+s+k;z,x,y]$$

$$\times {}_2F_1[\alpha+s+k+l,\ f+s+k+l;d+s+k+l;w], \tag{3.23}$$

$$K_{11}[\alpha,\alpha,\alpha,\alpha,c,\acute{c},\acute{\acute{c}},f;e,e,e,d;x,y,z,w]$$

$$=\sum_{s,k,l=0}^{\infty}\frac{(\alpha)_{s+k+l}(f)_{s+k+l}(c)_s(\acute{c})_k(\acute{\acute{c}})_l(xw)^s(yw)^k(zw)^l}{(e)_{s+k+l}(d)_{s+k+l}s!k!l!}$$

$$\times F_D^{(3)}[\alpha+s+k+l,c+s,\acute{c}+k,\acute{\acute{c}}+l;e+s+k;x,y,z]$$

$$\times {}_2F_1[\alpha+s+k+l,\ f+s+k+l;d+s+k+l;w], \tag{3.24}$$

where the decomposition formulas {(3.1), (3.2)}, (3.3), (3.4), (3.5), {(3.7), (3.8)}, {(3.9), (3.11), (3.12)}, (3.13), (3.14), (3.15), {(3.16)–(3.19)} and {(3.20)–(3.24)} follow from (2.3), (2.5), (2.7), (2.9), (2.12), (2.14), (2.17), (2.19), (2.21), (2.23) and (2.25) respectively. Note that the inverse series of each of the above direct series can be easily derived from the corresponding inverse formula in Section 2.

Next, the operator (1.17) can also be used for obtaining more decomposition formulas from our main results in Section 2. For instance, if in (2.7), we let

$$\Omega(m,n,p,q)=\frac{1}{(\acute{g})_n(h)_p},$$

multiply both sides by $t^{\lambda-1}u^{\gamma-1}$, replace $x,y,z$ and $w$ by $x\hat{D}[t_{(1)};u_{(1)}]$, $y\hat{D}[t_{(1)};u_{(0)}]$, $z\hat{D}[t_{(1)};u_{(0)}]$ and $w\hat{D}[t_{(1)};u_{(1)}]$ respectively and use the operator (1.17), then we find for Exton's quadruple functions $K_1$ and $K_{13}$ [6] that

$$K_1[\lambda,\lambda,\lambda,\lambda,\alpha,\alpha,\alpha,\delta;\gamma,g,h,\gamma;x,y,z,w]$$

$$=\sum_{s,k,l,r=0}^{\infty}\frac{(\lambda)_K(\alpha)_K}{(\gamma)_{s+k+r}(\acute{g})_{s+l+r}(h)_{k+l+r}s!k!l!r!}(xy)^s(xz)^k(yz)^l(-xyz)^r$$

$$\times K_{13}[\lambda+K,\lambda+K,\lambda+K,\lambda+K,\alpha+K,\delta,\alpha+K,\alpha+K;$$

$$\gamma+s+k+r,\gamma+s+k+r,\acute{g}+s+l+r,h+k+l+r;x,w,y,z]. \tag{3.25}$$

Similarly, starting from (2.17) and proceeding in the manner described above it is not difficult to obtain the following decomposition formulas involving the quadruple series $K_3$ of Exton [6] and $F_{19}^{(4)}$ and $F_{57}^{(4)}$ of Sharma and Parihar [14]:

$$K_1[\alpha,\alpha,\alpha,\alpha,\lambda,\lambda,\lambda,\acute{b};e,h,\acute{h},e;x,z,w,y]$$

$$=\sum_{s,k,l,r=0}^{\infty}\frac{(\lambda)_{2s+2k+l+r}(\alpha)_K(\acute{b})_{l+r}}{(e)_K(h)_{s+l}(\acute{h})_{k+r}s!k!l!r!}(xz)^s(xw)^k(yz)^l(yw)^r$$

$$\times F_{19}^{(4)}[\lambda+2s+2k+l+r,\lambda+2s+2k+l+r,\lambda+2s+2k+l+r,\acute{b}+l+r,$$

$$\alpha+K,\alpha+K,\alpha+K,\alpha+K;h+s+l,\acute{h}+k+r,e+K,e+K;z,w,x,y], \tag{3.26}$$

$$K_3[\alpha,\alpha,\alpha,\alpha,\lambda,\lambda,\delta,\delta;e,d,e,d;x,y,z,w]$$

$$=\sum_{s,k,l,r=0}^{\infty}\frac{(\lambda)_{2s+k+l}(\delta)_{2r+k+l}(\alpha)_K}{(e)_K(d)_Ks!k!l!r!}(xz)^s(xw)^k(yz)^l(yw)^r$$

$$\times F_{57}^{(4)}[\alpha+K,\alpha+K,\alpha+K,\alpha+K,\lambda+2s+2k+l,\delta+2r+k+l;$$

$$e+K,e+K,d+K,d+K;x,y,z,w]. \tag{3.27}$$

Further, by letting $e=\alpha$ in (3.26) and simplifying, we get

$$K_1[\alpha,\alpha,\alpha,\alpha,\lambda,\lambda,\lambda,\acute{b};\alpha,h,\acute{h},\alpha;x,z,w,y]$$

$$=(1-x)^{-\lambda}(1-y)^{-\acute{b}}\sum_{s,k,l,r=0}^{\infty}\frac{(\lambda)_{2s+2k+l+r}(\acute{b})_{l+r}}{(h)_{s+l}(\acute{h})_{k+r}s!k!l!r!}$$

$$\times\left(\frac{xz}{(1-x)^2}\right)^s\left(\frac{xw}{(1-x)^2}\right)^k\left(\frac{yz}{(1-x)(1-y)}\right)^l\left(\frac{yw}{(1-x)(1-y)}\right)^r$$

$$\times F_4\left[\alpha+K,\lambda+2s+2k+l+r;h+s+r,\acute{h}+k+r;\frac{z}{(1-x)},\frac{w}{(1-x)}\right], \tag{3.28}$$

$$
\begin{aligned}
&K_3[\alpha,\alpha,\alpha,\alpha,\lambda,\lambda,\delta,\delta;\alpha,d,\alpha,d;x,z,y,w] \\
&\quad = (1-x)^{-\lambda}(1-y)^{-\delta}\sum_{s,k,l,r=0}^{\infty}\frac{(\lambda)_{2s+k+l}(\delta)_{2r+k+l}}{(d)_K\, s!k!l!r!} \\
&\qquad \times \left(\frac{xz}{(1-x)^2}\right)^s\left(\frac{xw}{(1-x)(1-y)}\right)^k\left(\frac{yz}{(1-x)(1-y)}\right)^l\left(\frac{yw}{(1-y)^2}\right)^r \\
&\qquad \times F_1\left[\alpha+K,\lambda+2s+2k+l,\delta+2r+k+l;d+K;\frac{z}{(1-x)},\frac{w}{(1-y)}\right]
\end{aligned}
\tag{3.29}
$$

respectively.

## 4. Observations and concluding remarks

In the preceding sections, we have exploited a general procedure to deal with the symbolic operational images and the decomposition formulas of hypergeometric functions of two, three and four variables. The obtained results are proving the usefulness and the generality of the method which can easily be applied to other families of hypergeometric functions and polynomials. Also, it is important to note that the operator (1.17), the symbolic operational images and the particular decomposition formulas in the previous sections can, in turn, be used to state other needed properties of hypergeometric series. For instance, we discus here some of them.

### 4.1. Finite sums via operational images

One of the advantages offered by the use of the operational images (1.33) and (1.34) is the possibility of establishing finite sums for the double hypergeometric series $F^{(3)}[x,y,z]$. First, by employing (1.30) to (1.32) and taking advantage from the identity

$$
\begin{aligned}
&\bigl[1-(1-x\hat{M})\bigr]^m\bigl[1-(1-y\hat{N})\bigr]^n\bigl[1-(1-z\hat{R})\bigr]^p = x^m y^n z^p \hat{M}^m \hat{N}^n \hat{R}^p \\
&\quad (m,n,p \geqslant 0,1,2,\ldots),
\end{aligned}
$$

and definition (1.33), we can state that

$$
\begin{aligned}
&x^m y^n z^p \frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\prod_{j=1}^{B}(b_j)_{m+n}\prod_{j=1}^{\acute{B}}(\acute{b}_j)_{n+p}\prod_{j=1}^{\acute{\acute{B}}}(\acute{\acute{b}}_j)_{p+m}}{\prod_{j=1}^{E}(e_j)_{m+n+p}\prod_{j=1}^{G}(g_j)_{m+n}\prod_{j=1}^{\acute{G}}(\acute{g}_j)_{n+p}\prod_{j=1}^{\acute{\acute{G}}}(\acute{\acute{g}}_j)_{p+m}} \\
&\quad \times \frac{\prod_{j=1}^{C}(c_j)_m\prod_{j=1}^{\acute{C}}(\acute{c}_j)_n\prod_{j=1}^{\acute{\acute{C}}}(\acute{\acute{c}}_j)_p}{\prod_{j=1}^{H}(h_j)_m\prod_{j=1}^{\acute{H}}(\acute{h}_j)_n\prod_{j=1}^{\acute{\acute{H}}}(\acute{\acute{h}}_j)_p} = \sum_{s=0}^{m}\sum_{k=0}^{n}\sum_{r=0}^{p}(-1)^{s+k+r}\binom{m}{s}\binom{n}{k}\binom{p}{r} \\
&\quad \times F^{(3)}\left[\begin{matrix}(a_A)::(b_B);(\acute{b}_{\acute{B}});(\acute{\acute{b}}_{\acute{\acute{B}}}):-s,(c_C);-k,(\acute{c}_{\acute{C}});-r,(\acute{\acute{c}}_{\acute{\acute{C}}}); \\ (e_E)::(g_G);(\acute{g}_{\acute{G}});(\acute{\acute{g}}_{\acute{\acute{G}}}):\quad (h_H);\quad (\acute{h}_{\acute{H}});\quad (\acute{\acute{h}}_{\acute{\acute{H}}});\end{matrix}\; x,y,z\right].
\end{aligned}
\tag{4.1}
$$

Similarly, the use of the following identity

$$
\bigl[1-(1-x\hat{M}-y\hat{N}-z\hat{R})\bigr]^n = [x\hat{M}+y\hat{N}+z\hat{R}]^n \quad (n \geqslant 0,1,2,\ldots)
$$

allows to conclude that

$$
\begin{aligned}
&\frac{\prod_{j=1}^{A}(a_j)_n\prod_{j=1}^{B}(b_j)_n\prod_{j=1}^{\acute{\acute{B}}}(\acute{\acute{b}}_j)_n\prod_{j=1}^{C}(c_j)_n}{\prod_{j=1}^{E}(e_j)_n\prod_{j=1}^{G}(g_j)_n\prod_{j=1}^{\acute{\acute{G}}}(\acute{\acute{g}}_j)_n\prod_{j=1}^{H}(h_j)_n}x^n \\
&\quad \times F^{1+\acute{B}+H:\acute{C}+\acute{\acute{G}};\acute{\acute{C}}+G}_{C+\acute{G}+H:\acute{H}+\acute{\acute{B}};B+\acute{\acute{H}}}\left[\begin{matrix}-n,1-(h_H)-n,(\acute{b}_{\acute{B}}):(\acute{c}_{\acute{C}}),1-(\acute{\acute{g}}_{\acute{\acute{G}}})-n;(\acute{\acute{c}}_{\acute{\acute{C}}}),1-(g_G)-n; \\ 1-(c_C)-n,(\acute{g}_{\acute{G}}):(\acute{h}_{\acute{H}}),1-(\acute{\acute{b}}_{\acute{\acute{B}}})-n;(\acute{\acute{h}}_{\acute{\acute{H}}}),1-(b_B)-n;\end{matrix}\; -\frac{y}{x},-\frac{z}{x}\right] \\
&\quad = \sum_{r=0}^{n}(-1)^r\binom{n}{r}F^{(3)}\left[\begin{matrix}-r,(a_A)::(b_B);(\acute{b}_{\acute{B}});(\acute{\acute{b}}_{\acute{\acute{B}}}):(c_C);(\acute{c}_{\acute{C}});(\acute{\acute{c}}_{\acute{\acute{C}}}); \\ (e_E)::(g_G);(\acute{g}_{\acute{G}});(\acute{\acute{g}}_{\acute{\acute{G}}}):\quad (h_H);\quad (\acute{h}_{\acute{H}});\quad (\acute{\acute{h}}_{\acute{\acute{H}}});\end{matrix}\; x,y,z\right].
\end{aligned}
\tag{4.2}
$$

*4.2. Reduction formulas*

By exploiting the operational images (1.14), (1.25), (1.33)–(1.36) and the decomposition formulas of Section 2, we can derive the (known and new) reduction formulas for certain hypergeometric functions. First, in view of the composition rule

$$\left(1-x\hat{D}[t_{(1)};u_{(1)}]\right)^{-\beta_1-\cdots-\beta_n}=\left(1-x\hat{D}[t_{(1)};u_{(1)}]\right)^{-\beta_1}\cdots\left(1-x\hat{D}[t_{(1)};u_{(1)}]\right)^{-\beta_n}, \tag{4.3}$$

and the operational image (1.14), we easily find for Lauricella's function of $n$-variables [18, p. 33, (4)] $F_D^{(n)}$ that

$${}_2F_1[\alpha,\beta_1+\cdots+\beta_n;\gamma;x]=F_D^{(n)}[\alpha,\beta_1,\ldots,\beta_n;\gamma;x,\ldots,x], \tag{4.4}$$

which is a well-known reduction formula due to Lauricella (see e.g. [18, p. 34, Eq. (6)]).

Next, on letting $e=d=\alpha$ in (3.27) and simplifying, we get

$$\begin{aligned}&K_3[\alpha,\alpha,\alpha,\alpha,\lambda,\lambda,\delta,\delta;\alpha,\alpha,\alpha,\alpha;x,z,y,w]\\&\quad=(1-x-z)^{-\lambda}(1-y-w)^{-\delta}X_9\left[\lambda,\delta;\alpha;\frac{xz}{(1-x-z)^2},\frac{xyzw}{(1-x-z)(1-y-w)},\frac{yw}{(1-y-w)^2}\right],\end{aligned} \tag{4.5}$$

where $X_9$ is Exton's triple series [18, p. 86, (57a)].

Further, put

$$\Omega(m,n,p,q)=\frac{1}{(\alpha)_m(\beta)_n(\alpha)_p(\eta)_q}$$

in (2.3), multiply by $t^{\lambda-1}$, replace $x$, $y$, $z$ and $w$ by $x\hat{D}[t_{(1)};u_{(0)}]$, $y\hat{D}[t_{(1)};u_{(0)}]$, $z\hat{D}[t_{(1)};u_{(0)}]$ and $w\hat{D}[t_{(1)};u_{(0)}]$ respectively and apply the operator (1.17), thus having

$$\begin{aligned}&K_{10}[\lambda,\lambda,\lambda,\lambda,\alpha,\alpha,\beta,\delta;\alpha,\alpha,\gamma,\eta;x,z,w,y]\\&\quad=(1-x-z)^{-\lambda}X_8\left[\lambda,\beta,\delta;\alpha,\gamma,\eta;\frac{xz}{(1-x-z)^2},\frac{y}{(1-x-z)},\frac{w}{(1-x-z)}\right],\end{aligned} \tag{4.6}$$

where $X_8$ is Exton's triple series [18, p. 85, (41a)]. Similarly, a list is now given of some reduction formulas which may be obtained by the above mentioned method:

$$\begin{aligned}&K_9[\lambda,\lambda,\lambda,\lambda,\alpha,\alpha,\beta,\gamma;\alpha,\alpha,\gamma,\gamma;x,z,w,y]\\&\quad=(1-x-z)^{-\lambda}X_7\left[\lambda,\beta,\delta;\alpha,\gamma,\gamma;\frac{xz}{(1-x-z)^2},\frac{y}{(1-x-z)},\frac{w}{(1-x-z)}\right],\end{aligned} \tag{4.7}$$

$$\begin{aligned}&K_5[\lambda,\lambda,\lambda,\lambda,\alpha,\alpha,\beta,\beta;\alpha,\alpha,\beta,\beta;x,z,y,w]\\&\quad=(1-x-y-z-w)^{-\lambda}F_4\left[\frac{\lambda}{2},\frac{\lambda+1}{2};\alpha,\beta;\frac{4xz}{(1-x-y-z-w)},\frac{4yw}{(1-x-y-z-w)}\right],\end{aligned} \tag{4.8}$$

which is a known result [6, 118(4.1.1.4)];

$$\begin{aligned}&F_7^{(4)}[\lambda,\lambda,\mu,\mu,\alpha,\beta,\alpha,\beta;\alpha,\beta,\alpha,\beta;x,y,z,w]\\&\quad=(1-x-z)^{-\lambda}(1-x-z)^{-\mu}{}_2F_1\left[\frac{\lambda}{2},\frac{\lambda+1}{2};\alpha;\frac{4xz}{(1-x-z)^2}\right]{}_2F_1\left[\frac{\mu}{2},\frac{\mu+1}{2};\beta;\frac{4yw}{(1-y-w)^2}\right],\end{aligned} \tag{4.9}$$

$$\begin{aligned}&F_5^{(4)}[\lambda,\lambda,\lambda,\mu,\alpha,\alpha,\beta,\beta;\alpha,\alpha,\beta,\beta;x,y,z,w]\\&\quad=(1-x-y-z)^{-\lambda}(1-w)^{-\mu}H_4\left[\lambda,\mu;\alpha,\beta;\frac{xz}{(1-x-y-z)^2},\frac{yw}{(1-x-y-z)}\right],\end{aligned} \tag{4.10}$$

$$\begin{aligned}&F_{15}^{(4)}[\alpha,\alpha,\alpha,\beta,\lambda,\lambda,\mu,\lambda;\alpha,\alpha,\alpha,\gamma,;z,x,w,y]\\&\quad=(1-x-z)^{-\lambda}(1-w)^{-\mu}X_7\left[\lambda,\alpha,\beta;\gamma,\alpha;\frac{xz}{(1-x-z)^2},\frac{xw}{(1-x-z)(1-w)},\frac{y}{(1-x-z)}\right],\end{aligned} \tag{4.11}$$

where the formulas (4.7), {(4.8), (4.9), (4.10)} and (4.11) follow from (2.3), (2.5) and (2.11) respectively and $X_7$ is a triple series of Exton [18, p. 84, (43a)], $F_5^{(4)}$, $F_7^{(4)}$ and $F_{15}^{(4)}$ quadruple series of Sharma and Parihar [14] and $H_4$ is Horn's double series [18, p. 24, (12)].

Finally, let us stress that the schema suggested in Sections 1 and 2 can be applied to find symbolic operational images and decomposition formulas for other generalized hypergeometric functions. In a forthcoming papers we will consider the problems of establishing symbolic operational images and decomposition formulas for other generalized hypergeometric functions of Horn's type (see e.g. [18, Section 1.3]) and Lauricella's type of $n$-variables by following the technique discussed in this paper.

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